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The effect of nonlinear supratransmission in discrete structures: a review

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This paper provides an overview of studies on nonlinear supratransmission and related phenomena. This effect consists in the transfer of energy at frequencies not supported by the systems under consideration. The supratransmission does not depend on the integrability of the system, it is resistant to damping and various classes of boundary conditions. In addition, a nonlinear discrete medium, under certain general conditions imposed on the structure, can create instability due to external periodic influence. This instability is the generative process underlying the nonlinear supratransmission. This is possible when the system supports nonlinear modes of various nature, in particular, discrete breathers. Then the energy penetrates into the system as soon as the amplitude of the external harmonic excitation exceeds the maximum amplitude of the static breather of the same frequency.

The effect of nonlinear supratransmission is an important property of many discrete structures. A necessary condition for its existence is the discreteness and nonlinearity of the medium. Its manifestation in systems of various nature speaks of its fundamentality and significance. This review considers the main works that touch upon the issue of nonlinear supratransmission in various systems, mainly model ones.

Many teams of authors are studying this effect. First of all, these are models described by discrete equations, including sin-Gordon and the discrete Schrödinger equation. At the same time, the effect is not exclusively model and manifests itself in full-scale experiments in electrical circuits, in nonlinear chains of oscillators, as well as in metastable modular metastructures. There is a gradual complication of models, which leads to a deeper understanding of the phenomenon of supratransmission, and the transition to disordered structures and those with elements of chaos structures allows us to talk about a more subtle manifestation of this effect. Numerical asymptotic approaches make it possible to study nonlinear supratransmission in complex nonintegrable systems. The complication of all kinds of oscillators, both physical and electrical, is relevant for various real devices based on such systems, in particular, in the field of nano-objects and energy transport in them through the considered effect. Such systems include molecular and crystalline clusters and nanodevices. In the conclusion of the paper, the main trends in the research of nonlinear supratransmission are given.

Keywords: nonlinear supratransmission, soliton, discrete breather, nonlinear lattice dynamics, infra-transmission, solitary wave, computer model

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Эффект нелинейной супратрансмиссии в дискретных структурах: обзор

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В данной работе приводится обзор исследований, посвященных нелинейной супратрансмиссии и сопутствующим явлениям. Данный эффект заключается в передаче энергии на частотах, не поддерживаемых рассматриваемыми системами. Супратрансмиссия не зависит от интегрируемости системы, устойчива к демпфированию и различным классам граничных условий. Кроме того, нелинейная дискретная среда при некоторых общих условиях, накладываемых на структуру, может создавать неустойчивость, обусловленную внешним периодическим воздействием. Она является порождающим процессом, лежащим в основе нелинейной супратрансмиссии. Это возможно, когда система поддерживает нелинейные моды различной природы, в частности дискретные бризеры. Тогда энергия проникает в систему, как только амплитуда внешнего гармонического возбуждения превышает максимальную амплитуду статического бризера той же частоты.

Эффект нелинейной супратрансмиссии является важным свойством многих дискретных структур. Необходимыми условиями для его существования являются дискретность и нелинейность среды. Его проявление в системах различной природы говорит о его фундаментальности и значимости. В данном обзоре рассмотрены основные работы, затрагивающие вопрос нелинейной супратрансмисии в различных системах, преимущественно модельных.

Многими авторскими коллективами ведутся исследования данного эффекта. В первую очередь это модели, описываемые дискретными уравнениями, в том числе sin-Гордона и дискретным нелинейным уравнением Шрёдингера. При этом эффект не является исключительно модельным и проявляет себя в натурных экспериментах в электрических цепях, в нелинейных цепочках осцилляторов, а также в метастабильных модульных метаструктурах. Происходит поэтапное усложнение моделей, что приводит к более глубокому пониманию явления супратрансмиссии, а переход к разупорядоченным и с элементами хаоса структурам позволяет говорить о более тонком проявлении данного эффекта. Численные асимптотические подходы позволяют исследовать нелинейную супратрансмиссию в сложных неинтегрируемых системах. Усложнение всевозможных осцилляторов, как физических, так и электрических, актуально для различных реальных устройств, базирующихся на подобных системах. В том числе в области нанообъектов и транспорта энергии в них посредством рассматриваемого эффекта. К таким системам относятся молекулярные, кристаллические кластеры и наноустройства. В заключении работы приводятся основные тенденции исследований нелинейной супратрансмиссии.

Ключевые слова: нелинейная супратрансмиссия, солитон, дискретный бризер, нелинейная динамика решеток, инфратрансмиссия, уединенная волна, компьютерная модель

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1. Introduction

Nonlinear discrete structures can generate many different effects due to the nature of the connections of the particles of the system. This review considers the achievements of research on the effect of nonlinear supratransmission (NST). The essence of this effect is to transfer energy at frequencies not supported by the systems under consideration. The first publication devoted to this problem is the work of F. Geniet and J. Leon published in 2002 [Geniet, Leon, 2002], the foundations for which were laid a year earlier in [Caputo, Leon, 2001]. In this paper, a rule was formulated for excitation of waves in a discrete chain for energy transfer at frequencies outside the spectrum of the system. It consists in the fact that energy penetrates into the system as soon as the amplitude of the external harmonic excitation exceeds the maximum amplitude of the static breather of the same frequency.

Here, one should turn to the concept of a discrete breather (DB) for a more detailed understanding of the energy transfer mechanism. In the general case, DBs are understood as spatially localized, strictly periodic high-amplitude oscillatory modes in nonlinear discrete systems [Flach, Willis, 1998]. The existence of DBs as time-periodic solutions localized in a discrete space was first rigorously proven in [MacKay, Aubry, 1994] for infinite chains of locally diffusion-coupled nonlinear oscillators. This phenomenon has been the subject of numerous works covering theoretical, applied, and experimental aspects [Aubry, 1997; Flach, Gorbach, 2008; Manley, 2010; Zakharov, Korznikova, Dmitriev, 2017; Abdullina et al., 2019]. The variety of systems supporting DBs has led to a discussion of their contribution to various properties of discrete structures, including macroscopic ones [Захаров и др., 2016; Savin, Korznikova, Dmitriev, 2022; Захаров и др., 2017]. For example, one of the latest works shows the contribution of breathers to the heat capacity of crystals [Singh et al., 2021].

There are many different types of DB. Discrete breathers can be classified according to a number of features: size factor, symmetry, mobility, type of nonlinearity, etc. For example, zero-dimensional, one-dimensional, two-dimensional and three-dimensional excitations. It is possible to obtain both stationary and moving discrete breathers [Cuevas-Maraver, Chacon, Palmero, 2016]. According to the type of nonlinearity, one distinguishes between DBs with hard and soft types. The first includes excitations, when the oscillation frequency increases with the amplitude. Such discrete breathers can be detected in the form of modes above the optical branch of the phonon spectrum, or when high-amplitude vibrations of the heavy lattice component are excited, and the vibration frequency is split off from the acoustic branch into the band gap of the spectrum [Korznikova et al., 2020].

In the context of the problems under consideration, it is necessary to speak about discrete breathers in complex systems, both in real and model ones, with some reservation. The fact is that an object strictly periodic in time is obtained in numerical simulation or a full-scale experiment only in the case of ideal adjustment of the initial conditions of the Cauchy problem to a certain manifold of low dimension in a multidimensional space of all possible initial values of the coordinates of individual particles and their velocities [Chechin, Dzhelauhova, Mehonoshina, 2006]. Such a fine tuning is difficult to implement even in a computational experiment. Moreover, it is practically impossible to do this when setting up any physical experiments, especially in cases where breather-like objects arise spontaneously [Ryabov et al., 2020; Shepelev et al., 2020; Sato, Hubbard, Sievers, 2006]. More details about the nature of discrete breathers can be found in review papers on this topic, for example, in the work of Dmitriev et al. [Дмитриев и др., 2016].

Thus, F. Geniet and J. Leon emphasize the role of localized excitations in the transfer of energy into the depth of the system from the source of periodic action. Developing this topic in his subsequent works [Geniet, Leon, 2003; Leon, 2003; Leon, Spire, 2004; Leon, 2007], the authors conclude that the revealed effect is universal and can be extended to any discrete systems with a forbidden frequency band of various nature. The supratransmission does not depend on the integrability of the system, it is resistant to damping and various classes of boundary conditions. In addition, a nonlinear discrete medium, under certain general conditions imposed on the structure of the nonlinearity, can create an instability of the damped profile due to an external periodic action. This instability is the generative process underlying the nonlinear supratransmission. This is possible when the system supports nonlinear modes of various nature.

A more extended approach to nonlinear discrete structures was carried out in the work on numerical methods with consistency properties in the energy domain for a class of dissipative nonlinear wave equations with applications to the Dirichlet boundary value problem [Macías-Díaz, 2009]. It is shown here that the process of nonlinear supratransmission is absent in media described by undamped radially symmetric sine-Gordon equations, thus proving that not every nonlinear system with a bandgap can support this process. Further, a generalization to N-dimensional nonlinear equations is made, which in various ways express a quantitative model that describes discrete arrays consisting of coupled harmonic oscillators. By using nonlinear supratransmission, controlled propagation of wave signals can be achieved. With the development of this direction, nonlinear systems of the most diverse nature were studied, a large number of models and theories were built, which we will turn to in the following sections of this review.

2. Mathematical models and the essence of the supratransmission effect for various discrete systems

The first models were the simplest one-dimensional chains of oscillators. The work [Geniet, Leon, 2002] considers the sin-Gordon chain:

$$\ddot{u}_n - c^2(u_{n+1} - 2u_n + u_{n-1}) + \sin u_n = 0. \tag{1}$$

The following boundary conditions were imposed: $n > 0$ (i. e., a semi-infinite straight line), $u_0(t)$ is the harmonic function of the external action on the edge of the oscillator chain, $u_n(0)$ are the initial coordinates and $\dot{u}_n(0)$ is the velocity. Under these conditions, the equation has an exact stationary solution in the form of a breather, found using the inverse scattering problem [Захаров и др., 1980]:

$$u = 4 \arctan\left(\frac{\sqrt{1-w^2}\cos(wt)}{w\cosh\left(\sqrt{1-w^2}x\right)}\right), \tag{2}$$

for which the dispersion relation will have the form: $w^2 = 1 + 2c^2(1 - \cos(k))$.

By varying the initial conditions, it is possible to achieve a situation in which energy transfer occurs at forbidden frequencies for a given system. It is important to note that discrete breathers accumulate the energy of external action. The occurrence of the effect of supratransmission is possible at amplitude values exceeding a certain threshold value.

Similar results were obtained for the nonlinear Klein – Gordon chain, where the system is described by the Hamiltonian

$$H_n = \frac{1}{2}\dot{u}_n^2 + \frac{c^2}{2}(u_{n+1} - u_n)^2 + V(u_n). \tag{3}$$

Here, the authors of [Geniet, Leon, 2002; Geniet, Leon, 2003] have shown that there is a supratransmission effect, despite the fact that the model does not have an exact solution in the form of a discrete breather. In [Leon, 2003], as the reasons for the manifestation of this effect, they speak of an unstable state generated by an external periodic action. This instability is the generative process underlying the nonlinear supratransmission. In this paper, we consider a number of models based on the exponential potential and the Eckart potential, as well as the previously considered sin-Gordon model. Both in this work and in the subsequent one [Leon, Spire, 2004], the discussion of the appearance of instability and, as a consequence, the appearance of a soliton-like perturbation continues.

In 2021, for the first time, the NST phenomenon was shown to be present in time-fractional damped sine-Gordon equations. However, completely in the fractional version of the wave that passed into the medium, it attenuates even when the NST effect appears [Macías-Díaz, 2021; Macías-Díaz, Bountis, 2021]. For a model similar in meaning [Pechac, Frazier, 2021], we numerically study the NST phenomenon in an active nonlinear system modeled by a one-dimensional/two-dimensional discrete sine-Gordon equation with nonlocal feedback. The authors found that two modes can manifest themselves. One with a single threshold in amplitude, the second, in the case of nonlocal feedback, shows the presence of several thresholds in amplitude for the manifestation of supratransmission.

The conclusions made in [Geniet, Leon, 2003; Leon, 2003; Leon, Spire, 2004; Leon, 2007] led to models based on the discrete nonlinear Schrödinger equation [Khomeriki, 2004; Susantot, 2008; Tchinang Tchameu, Tchawoua, Togueu Motcheyo, 2016], as well as to the prospects for the practical application of this effect in optical waveguides. It is shown that the array of optical waveguides can become transparent to the beam if its intensity exceeds a certain threshold value. Let us consider in more detail the results obtained for this case. Let's write the Schrödinger equation

$$i\frac{\partial \psi_j}{\partial z} + \psi_{j+1} + \psi_{j-1} + 2|\psi_j|^2\psi_j = 0, \quad \psi_0 = Ae^{i\Delta z}, \tag{4}$$

where z is time, A is the amplitude, Δ is the frequency of the external action [Khomeriki, 2004].

Modeling the proposed system, varying the parameters of the external influence [Khomeriki, 2004], the corresponding dependencies were obtained, indicating several possible scenarios for the evolution of the system. If the frequency of exposure was not included in the forbidden region, then the perturbation propagated in the system, gradually fading away. In the case of the frequency of the forbidden region, but not at a sufficiently large amplitude, discrete breathers were excited in the vicinity of the external action, without penetrating into the remote regions of the array of optical waveguides. Thus, the effect of supra-transmission through the band gap exists if there are dynamic localized solutions in the system. These objects can be moving discrete breathers [Khomeriki, 2004; Mandelik et al., 2003].

The complication of models of physical systems, on the one hand, leads to the impossibility of an analytical search for a solution that would make it possible to explicitly predict and, using mathematical modeling, study the NST effect for such systems, on the other hand, more complex systems include a greater variety of forms and features of nonlinear effects. The work [Anghel-Vasilescu et al., 2010] also considers the Schrödinger equation. Its example shows the effectiveness of the proposed asymptotic method for studying the properties of supratransmission and solitons. This made it possible to solve the problem of determining the NST threshold in nonintegrable N-component systems by obtaining an asymptotic solution based on a linear damped soliton profile, which depends on N parameters, as well as by zeroing the Jacobian for such a system. Developing this approach in [Yu, Wang, Tao, 2011], the authors propose a method for generating and studying the dynamics of spatial light solitons in a birefringent medium with a quadratic nonlinearity. As an example, a $LiGaTe_2$ crystal is considered. The generation of a birefringent soliton appears as a special nonlinear effect that allows light to penetrate and propagate in the form of localized beams. In [Togueu Motcheyo et al., 2017], it was shown for the first time that the supratransmission threshold can be found for a discrete nonlinear Schrödinger equation modeling optical waveguide arrays with Kerr nonlinearity by using the 2D map approach. Recent studies of systems based on the discrete Schrödinger equation are devoted to studying the formation of dark solitons [Togueu Motcheyo et al., 2019]. The possibility of creating devices for generating discrete slot solitons is emphasized.

The next important model is the consideration of Josephson junctions. At the moment, the study of the effect of energy supertransfer in such systems is considered in detail in three works [Chevriaux, Khomeriki, Leon, 2006; Macías-Díaz, Puri, 2008b; Macías-Díaz, 2017b]. Let us consider in detail the model of [Macías-Díaz, Puri, 2008b]. It studies the differential equation, which is a variation of equation (1):

$$\ddot{u}_n - c^2 \Delta u_n + \gamma_n \dot{u}_n + \sin u_n = \mu, \tag{5}$$

where it is assumed that γ and c are positive real numbers. At its core, γ is an external damping coefficient, and c is called the coupling coefficient. The model contains N Josephson junctions, which are subject to the boundary conditions $u_n(0) = 0$, $\frac{du_n}{dt}(0) = 0$ for $1 \leqslant n \leqslant N$ and $u_0 - u_1 = \frac{\phi(t)}{c^2}$, $u_{n-1} - u_N = 0$ for $t \in (0, +\infty)$. At the same time, $\gamma_n = \gamma$ for $n < N$, $\gamma_N = \gamma + \frac{1}{R}$. R is output impedance. The value of N is large enough to represent the array of contacts as infinite. The numerical solution was carried out by two different methods, which gave good agreement. In addition, an analytical expression is obtained for the minimum amplitude at which the NST effect manifests itself: $A_s = 2c\left(1 - \omega^2\right)$. The authors also discovered infratransmission, the possibility of which was first predicted in [Chevriaux, Khomeriki, Leon, 2006], which in fact represents the transfer of energy into the system at amplitudes somewhat lower than for supratransmission. According to the data obtained for the system under consideration, the difference was about 28 % [Macías-Díaz, Puri, 2008c].

Further development of studies on the effect of supratransmission in Josephson junctions is presented in a 2017 paper [Macías-Díaz, 2017a; Piña-Villalpando, Macías-Díaz, Kurmyshev, 2019], where a generalization is made using discrete Riesz differential operators. The simulation revealed the presence of nonlinear hysteresis loops for various noninteger orders of the fractional derivative. This shows that the nonlinear bistable mode in the classical Josephson transmission line is preserved in the presence of fractional orders. As a mathematical application, the authors of [Macías-Díaz, 2017b] suggest using the hysteresis phenomenon for controlled signal propagation in fractional Josephson lines. One of the latest studies in this direction is devoted to the generation of breathers using controlled magnetic pulses [De Santis et al., 2022].

An important task was the problem of transition to two spatial variables with Neumann or Dirichlet boundary conditions. It is discussed with some variations in a number of works by Macías-Díaz et al. [Macías-Díaz, 2008b; Macías-Díaz, 2008a; Macías-Díaz, 2008c; Macías-Díaz, Medina-Ramírez, Puri, 2009; Macías-Díaz, Ruiz-Ramírez, Flores-Oropeza, 2009; Macías-Díaz, Puri, 2008a; Macías-Díaz, 2009; Ruiz-Ramírez, Macías-Díaz, 2010; Macías-Díaz, Jerez-Galiano, 2010; Macías-Díaz, 2011; Macías-Díaz, 2017a]. On the whole, it is in full agreement with the one-dimensional case. We emphasize that an article [Bountis, 2020] was published in 2020 in which the results of a number of studies for one-dimensional cases are structured, generalizing the results of many researchers. The author says that it is still necessary to carry out a local and global analysis of the stability of simple periodic oscillations in order to identify the regions of energy and parameters associated with globally stable motion, as well as regimes of strong and weak chaos.

The search for interpretations of the mechanisms led to the study of systems with more specific conditions, which expanded the variety of manifestations of this effect. In [Bodo et al., 2009], equation (1) is considered for amplitudes below the threshold values for the occurrence of supratransmission. However, a result was obtained indicating the possibility of the occurrence of moving discrete breathers in the presence of white noise. It is shown that noise can trigger the supratransmission effect below its deterministic threshold with a set probability. After analyzing various noise parameters, it was found that there is an appropriate noise level that ensures the existence of breather modes with the best coherence [Bodo et al., 2010; Bodo, Morfu, 2013]. In the modified Klein – Gordon and fifth-order sine-Gordon models, including both the classical linear connection and the nonlinear connection, their contribution to this effect was studied [Alima et al., 2017; Malishava, Khomeriki, 2015].

The possibility of energy transfer over considerable distances in model systems and the implementation of this process through localized standstills in the form of soliton waves or dynamic discrete breathers led to the idea of the possibility of heat transfer from a less heated area to a hotter one through the phenomenon of supratransmission [Ai, He, Hu, 2010]. The first model was a Frenkel – Kontorova chain, on which the boundary conditions characteristic of the supratransmission effect were imposed. The system in this case was described by the Hamiltonian

$$H = \sum_i \frac{p_i^2}{2m} + \frac{1}{2}k(x_i - x_{i+1} - a)^2 - \frac{V}{(2\pi)^2}\cos(2\pi x_i) - \delta x_i F(t), \tag{6}$$

where x_i and p_i are, respectively, the position and momentum of the particle, m is the mass of the particle, k is the coupling constant, a is the equilibrium distance between the particles, V is the amplitude of the local potential, δ is the Dirac delta function. The external harmonic effect was given by the function $F(t) = A_0 \sin(wt)$, where A_0 is the amplitude, and w is the frequency. The authors observed the occurrence of a resonant phenomenon of heat flow. It has been established that there is a value of the excitation frequency at which the heat flux takes on a maximum value. Naturally, the value of this frequency is determined by the parameters of the model.

The most important conclusion in the work is that heat can be pumped from a low-temperature region to a high-temperature one, by appropriately adjusting the external influence on the system. In contrast to other models, it is demonstrated here that there is no amplitude threshold for the NST effect in a thermostat.

Continuing the question of crystal lattices, it was shown in [Yu, Wang, Tao, 2011] that a nonlinear resonance on impurities makes a localized excitation (usually a discrete breather) unstable, which leads to the emission of solitons, which looks completely different than in the case of direct excitation of homogeneous nonlinear systems without impurities. This approach makes it possible to create solitons at small values of the force of a harmonic external action. Also, a feature of such excitations is the low speed of their propagation, which can be controlled by the parameters of external influence.

Models of electrical circuits have also led to the discovery of the effect of nonlinear supratransmission in them. Such model systems support a wide range of phenomena. In particular, in [Kenmogne et al., 2015], using the exact discrete equations of the network and the extended nonlinear Schrödinger equation, the generation of solitons was discovered, the threshold value of the signal amplitude was calculated, at which a pulsed soliton is emitted into the network. Such systems will be considered in more detail below, since they lend themselves well to natural experiment.

Further development of models with electrical circuits is presented in [Zheng et al., 2018; Zheng et al., 2019; Mosquera-Sánchez, De Marqui, 2021], where a piezoelectric metamaterial integrated with bistable circuits is considered. The bistable circuit can be described in this case by the expression

$$LC_p\ddot{U} + RC_p\dot{U} + P(U) = 0, \tag{7}$$

where P is a function of stress and can be explicitly given as $P(U) = U - gV_d$, where g is the gain of a single negative feedback op-amp, and can be calculated as $g = 1 + \frac{R_2}{R_1}$. Expression (7) is equivalent to the equation of a mechanical oscillator. In this case, the equilibrium stress can be calculated from the condition $P(U) = 0$. Complementing this expression with equations for describing the dynamics of the piezo-metal material model, one can obtain a system of equations describing the elementary section of this circuit. The final expressions for computer analysis are too cumbersome to give them in this review with explanations; the reader can find them in the study [Zheng et al., 2019].

The use of electrical circuits as equivalents in many systems is an effective tool. So in [Ndjomatchoua et al., 2016], microtubules, which are important biological molecules of living organisms, are studied using this approach. They play an important role in cell division, the transport of organelles within cells, and the processing of electrical signals in neurons. The model is a discrete circuit including a cubic nonlinear resistance. Nonlinear resistance was used to represent the ion flow through the nanopores of microtubules. In differential form, the equation describing the dynamics of the system can be represented as follows:

$$\ddot{V}_n + \omega_0 \left(3LB_1 V_n^2 + (rC_0 - LB_2)\right) \dot{V}_n + rB_1 V_n^3 - rB_2 V_n = V_{n-1} - 2V_n + V_{n+1}. \tag{8}$$

This expression is written taking into account Kirchhoff's laws. The factors B_1 and B_2 represent the nonlinear and linear coefficients, and the terms containing them describe the contribution of the ion flux introduced by the nonlinear resistance. In turn, $\omega_0 = \sqrt{\frac{1}{LC_0}}$, the time is defined dimensionless $\tau = \omega_0 t$, and the external influence was determined by the harmonic function $V_0 = U\cos(w\tau)$. The system is considered both with and without dissipation. As a result, infratransmission and supratransmission were discovered; these nonlinear effects can quite well describe the processes observed in natural experiments. The authors emphasize, that from a biological point of view, a certain intensity of the ion flow through the nanopores of microtubules can provide a high frequency of processing electrical signals by neurons. In addition, infratransmission in microtubules of neurons can be used to explain why repeated stimulation with a high-frequency magnetic field is able to reduce the motor signs of Parkinson's disease.

The essence of the NST effect is revealed in discrete structures in violation of periodicity. The generation of localized states occurs due to a combination of two factors: dispersion and nonlinearity. In [Yousefzadeh, Phani, 2016], disorder is introduced into the structure due to small changes in stiffness parameters taken from a homogeneous statistical distribution. The system itself was a nonlinear periodic structure of finite length with weakly connected nodes. The authors showed that, although in some cases the amplitude threshold for the NST differs, in general, this value is robust to disorder when averaged over the entire ensemble. For harmonic excitation far from the edge of the allowed frequencies of the system, the increase in disorder has little effect on the transmitted energy.

3. The effect of supratransmission in molecular dynamics models

In this section, we consider models based on the molecular dynamics method. The application of this approach makes it possible to study many nonlinear effects in realistic structures, for example, in the crystal lattices of real alloys. The study of a two-dimensional model of the Pt_3Al alloy [Медведев и др., 2011] can be attributed to the first work in this direction. The interaction of particles was described by the Morse pair potential: $\varphi(r_{ij}) = D\beta e^{-\alpha r_{ij}}\left(e^{-\alpha r_{ij}} - 2\right)$, where D is the energy parameter corresponding to the depth of the potential well, α is the parameter that determines the rigidity of interatomic bonds, r_0 is some averaged equilibrium distance over the coordination spheres, in which the interaction between atoms is taken into account.

However, in this work, the NST effect was not explicitly distinguished. In this case, the excitation of discrete breathers near the impact zone is shown, which was carried out as follows: $u[i] = u_x[i] + u_0 \sin(0.5w_b t)^2$, where $u_x[i]$ is the horizontal component of the speed of particles with number i. As a result, periodic excitation of discrete breathers near the impact zone was recorded. The periodicity of the process indicates that the energy was transferred to the computational cell by means of the NST. A return to the 3D model of this alloy was carried out in [Чередниченко и др., 2019]. In this case, the computational cell contained $3 \cdot 10^5$ particles interacting through the potential obtained by the embedded atom method (EAM-potential), for which the total energy E of the crystal can be expressed as $E = \frac{1}{2}\sum_{i,j,\, i\neq j}\varphi_{ij}(r_{ij}) + \sum_i F_i(\rho_i)$, where φ_{ij} represents the pair energy between atoms i and j separated from each other by a distance r_{ij}, and F_i is the energy associated with an embedded atom i at

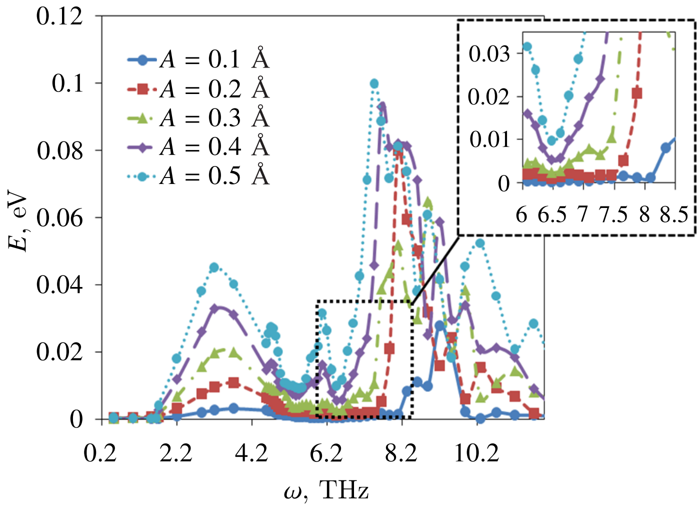

Figure 1. Dependence of the energy absorbed by the computational cell per atom per picosecond on the frequency of the external action and the amplitude [Чередниченко и др., 2019]

a local location with electron density ρ_i. The electron density can be calculated using the formula $\rho_i = \sum_{j,\, j\neq i} f_j(r_{ij})$, where $f_j(r_{ij})$ is the electron density in the area of atom i located at a distance r_{ij} from atom j. Periodic exposure was carried out according to the following laws: $Z_1(t) = A\sin(wt)$, $Z_2(t) = A(\sin(wt))^2$ and $Z_3(t) = A|\sin(wt)|$ along the Z axis with frequencies close to the natural frequencies of the breathers (6–8 THz), as well as with amplitudes from 0.2 to 0.3 Å. Next, the energy absorption by the remaining part of the cell was analyzed. For an example, Fig. 1 shows the dependence for the harmonic function. Thus, the result obtained for the three-dimensional case was similar to that obtained for the two-dimensional one. A detailed consideration of the mechanism of energy penetration into the depths of the crystal was carried out in [Чередниченко и др., 2019].

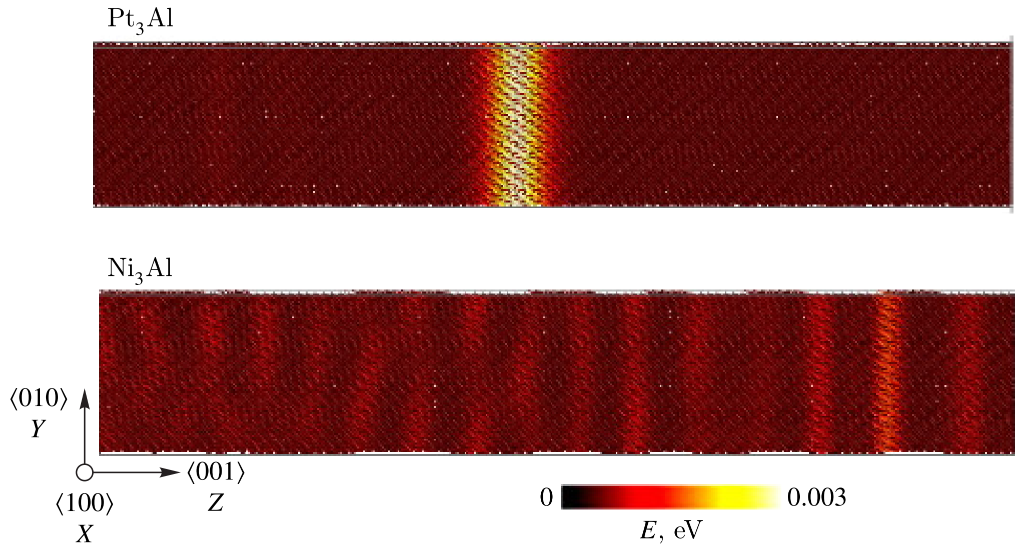

Figure 2. Distribution of kinetic energy in crystals Pt_3Al and Ni_3Al

As already noted, as a result of the action, discrete breathers were excited near the crystal surface, which, in turn, gave rise to local instability, which was transformed into solitary waves capable of moving along the crystal. To demonstrate that the NST effect is possible in the presence of a band gap in the crystal, a model Ni_3Al crystal was analyzed. A comparison is shown in Fig. 2. As a result, it is possible to observe the appearance of solitary waves in the phonon spectrum for a crystal with a band gap. In this case, the number of excited waves depends on the time and parameters of the impact.

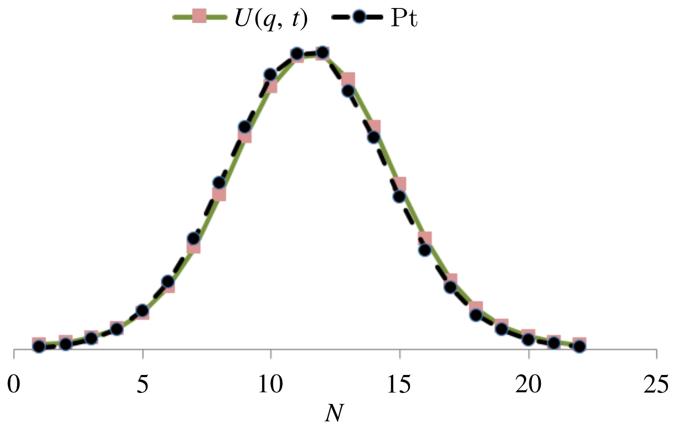

Figure 3. Soliton profile in Pt_3Al and the profile described by function (9). The amplitude is plotted along the vertical axis, and the number of the atom in the row is plotted along the horizontal axis

Such a solitary wave, which has a bell-shaped form (Fig. 3), is a soliton, because it is nonlinear, propagates at a constant speed, and does not interact with other waves when colliding with them [Захаров, Дмитриев, Корзникова, 2021]. In the theory of solitary waves in solids [Аэро, Булыгин, 2009], the following solution of the sine-Gordon equation of the form is known:

$$U(x, t) = \frac{2\alpha^2 \cosh \varepsilon^2}{1 + \alpha^2 \cosh \varepsilon^2}, \tag{9}$$

where $\varepsilon = x - Vt$, α is the amplitude multiplier, x is the coordinate, V is the speed of the soliton, t is the time. By selecting the appropriate values of the parameters of this solution, the profile of a solitary wave is obtained, which is in good agreement with the profile of the resulting soliton in a Pt_3Al crystal.

In [Захаров и др., 2019] it is shown that such solitary waves can propagate thousands of nanometers deep into a defect-free crystal without changing shape and speed. This mechanism of energy transport through the crystal, by means of solitary waves, seems to be one of the most efficient, and the mechanism for generating such waves is relatively simple.

The study of the dynamics of such waves and their interaction with bulk defects in crystals is studied in [Zakharov et al., 2020; Yousefzadeh, Phani, 2016]. It has been established that the degree of energy dissipation by dislocations depends on the mutual orientation of the wave front and the extraplane of the dislocation. It was shown that the waves under consideration are able to overcome obstacles whose linear dimensions are greater than their length. The energy of such waves is sufficient for structural-energy transformations during interaction with the crystal defects under consideration.

Noteworthy is the molecular dynamics study of the excitation of discrete breathers and the NST effect in a graphene nanoribbon [Evazzade et al., 2017]. Uniform deformation of graphene can lead to the appearance of a gap in its phonon spectrum, which makes it possible for gap DBs to exist. The authors obtained a result that indicates the absence of an external influence amplitude threshold for the manifestation of supratransmission. The linear regime was considered, i. e., with relatively small amplitudes at DB frequencies. In this case, a frequency range with nonzero power is detected. The explanation for this effect lies in the fact that such a harmonic shift causes the excitation of DBs at the interface between the action and the nanoribbon. This leads to the emission of a low-frequency phonon due to local thermal expansion.

4. Applied aspects of NST and experimental research

Almost every work, including the first pioneering work, touching upon the issue of nonlinear supratransmission, assumed practical implementation and application in the scientific or technical field. One of the first applied aspects of this effect is the construction of sensitive detectors of ultraweak signals based on the matrix of Josephson junctions [Chevriaux, Khomeriki, Leon, 2006]. The authors of the work proposed a scheme of such a detector. The work is based on a complete analytical analysis of bistability in a long Josephson junction and in an array of short junctions. For the first time, a tool has been obtained that allows one to calculate the threshold amplitude at which a device must operate in order to become an ultrasensitive detector. [Macías-Díaz, Puri, 2008b] proved that it is possible to transfer binary information in discrete arrays of Josephson junctions using the nonlinear supertransmission and infratransmission processes. And in [Macías-Díaz, 2017b] this was proved for controlled signal propagation in fractional Josephson transmission lines. In the absence of dispersive and dissipative effects, the model showed high reliability for sufficiently long periods of generation of single-bit signals, regardless of the distance between the sources [Macías-Díaz, Puri, 2007]. A similar approach is considered in [Macías-Díaz, 2010; Macías-Díaz, 2008a], where the $(2 + 1)$-dimensional Frenkel – Kontorova model was studied.

For systems described by the sine-Gordon equation, the effect can be used for signal processing purposes to improve the detection of weak noisy signals, thereby building detectors that take into account the noise [Bodo et al., 2009]. Further, in [Bodo et al., 2010], the authors continued the study for an electrical circuit controlled by the fifth-order Klein – Gordon equation and experimentally showed the possibility of signal transmission at forbidden frequencies of this model.

An interesting and promising application of the NST effect can be the development of a heat pump. The idea was proposed in [Ai, He, Hu, 2010], which we have already discussed. A feature of practical application is that the effect of thermal resonance does not depend on the size of the systems under consideration. Therefore, one would expect further developments in relation to thermal resonance and pumping in various physical systems, for example, in granular systems. One of the latest works [Cui et al., 2022] is devoted to such objects. It discovered the effect of NST, as well as 4 more modes of energy transfer. It is emphasized that the results obtained can be useful in the design of mechanical or other types of flow control. Research is also ongoing on physical pendulums of a more complex configuration, in the form of coupled pairs of chains of pendulums [Kamdoum Kuitche et al., 2022]. The authors used both an experimental technique and a numerical experiment. An even more complex configuration is discussed in [Adile et al., 2021], with an integrated approach to the study of various modes of a mechanical system, up to the construction of a nonlinear Schrödinger equation using the perturbation method. The equation is used to search for modulated impulsive and dark solitons as approximate solutions to the network equation.

An important place in the discussion of systems supporting nonlinear supratransmission is occupied by metastable modular metastructures. The authors of [Wu, Wang, 2018; Liu, Cai, Wang, 2019; Wu, Wang, 2019; Zhang, Fang, Xu, 2020], numerically and analytically investigate them. It is shown that in the case of bifurcation instability, the energy transfer coefficient can increase sharply when the input amplitude exceeds a certain threshold. By intelligently integrating the supertransmission property into a nonlinear periodic chain with spatial asymmetry, the metastable structure is able to realize nonreciprocal wave propagation due to the NST effect depending on the direction along the chain. In this paper, the authors proposed an efficient methodology with great potential for designing systems with desired adaptable characteristics of nonreciprocal propagation of wave energy, using the concept of a reconfigurable metastable modular metastructure.

Electrical nonlinear systems are the simplest to implement and amenable to simple modeling. A number of researchers drew attention to the possibility of the existence and prospects for the manifestation of NST in such systems [Tao et al., 2012a; Tao et al., 2012b; Togueu Motcheyo et al., 2013; Kenmogne et al., 2015; Koon, Marquié, Dinda, 2014; Togueu Motcheyo, Tchawoua, Tchinang Tchameu, 2013].

For example, in [Koon, Marquié, Dinda, 2014], a 200-element electric line is studied, in which solitons are generated and propagate due to the formation of a modulation instability at the boundaries. The authors note that the resulting solitons are greatly hindered by dissipation, which occurs due to the imperfection of the electric line components (inductances). However, dissipation does not prevent the emergence of the process of modulation instability, but strongly affects its development and makes it difficult to strictly experimentally measure the characteristics of the obtained solitons. In particular, the authors showed that dissipation prevents the generation of solitons when the excitation voltages are close to the theoretical NST threshold in the model without it. Accordingly, this imposes restrictions on the possibility of generating solitons; in order to obtain a clear sequence of solitons, it is necessary to use components of very high quality in terms of dissipation. The importance of these studies lies in the fact that it was possible to obtain experimental confirmation of the NST effect in electrical circuits consisting of only two components: a capacitor and a coil.

The theoretical model of a piezoelectric metamaterial with bistable circuits considered above was implemented in a full-scale experiment in the same work [Zheng et al., 2019]. Unlike nonlinear mechanical metamaterials, the nonlinear properties of bistable circuits in the proposed piezo-metastructure can be easily controlled by adjusting constant voltage sources or changing the gains of negative feedback operational amplifiers.

The possibility of flexible configuration can be used to organize adaptive nonreciprocal signaling. In general, the prospects of such systems considered here open up new possibilities for controlling the transmission of elastic waves.

5. Conclusion

The effect of nonlinear supratransmission is an important property of many discrete structures. A necessary condition for its existence is the discreteness and nonlinearity of the medium. Its manifestation in systems of various nature speaks of its fundamentality and significance. This review considers the main works that touch upon the issue of nonlinear supratransmission in various systems, mainly model ones.

Based on the retrospective of the study of this phenomenon described above and the results obtained, we can say that an active search is underway for the practical application of this effect for various purposes, including scientific ones. The gradual complication of models leads to a deeper understanding of the supratransmission phenomenon, and the transition to disordered structures with elements of chaos allows us to speak of a more subtle manifestation of this effect.

A number of promising directions for further research should be identified. These areas primarily include systems that do not have an exact analytical solution, with the complication of conditions both for the system itself and for external influences. However, numerical asymptotic approaches make it possible to study NSTs in them. The study of various physical oscillators will also continue.This, of course, relevant for various real structures based on such systems. Experimental studies of electrical circuits of various compositions should be attributed to the same direction. An important direction is research in the field of nanoobjects and energy transport in them through NST. These are various molecular and crystalline systems. All this is united by a common trend towards the search for mechanisms to control energy flows in its various manifestations.

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